10th Week Friday

28 Nisan 2023 Cuma 10:26

Vector Spaces → Subspaces

$(V, \oplus, \odot)$

Countries with special citizens

Bases → not unique.

for V.S.

↳ Skeletons for citizens

$\dim(V) = n$

$B = \{v_1, v_2, \dots, v_n\}$

$3v_1 + 5v_2 + \dots + (-2)v_n \rightarrow$ Ayşegül

$-2v_1 + 0v_2 + \dots + 7v_n \rightarrow$ Halil

⋮

$C = \begin{bmatrix} | & | & \cdots & | \\ w_1 & w_2 & \cdots & w_n \\ | & | & & | \end{bmatrix}_{n \times n}$

C, D, …

$\{w_1, w_2, \dots, w_n\}$

Ayşegül : $\alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$

$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = [\text{Ayşegül}]_C$

$\begin{bmatrix} 3 \\ 5 \\ \vdots \\ -2 \end{bmatrix} = [\text{Ayşegül}]_B$ ✓

$[v]_C$

$v =$ Ayşegül

$= E\,[\text{Ayşegül}]_E = C\,[\text{Ayşegül}]_C = B\,[\text{Ayşegül}]_B = \dots$

E → identity; $[\ ]_E$ → standard basis.

C ✓, $[\text{Ayşegül}]_C$ ✓, B ✓, $[\text{Ayşegül}]_B$?

$B^{-1}C\,[\text{Ayş.}]_C = [\text{Ayş.}]_B$

↳ The transition matrix from C to B.

Linear Transformations

On sets we define functions

f : $A \xrightarrow{\text{mapping}} B$ (A domain, B range)

$a \in A \mapsto b \in B$

$\forall x \in A \quad f(x) = 2x + 5 \in B$

→ $L : V \longrightarrow W$

V: vector space $(V, \oplus, \odot)$

W: vector space $(W, \boxplus, \boxdot)$

$v \in V \overset{\text{map to}}{\longmapsto} w \in W$ ⇝ an operation

$v_1, v_2 \in V$

? ✓ 1) LHS $L(v_1 \oplus v_2) = \underbrace{L(v_1)}_{\in W} \boxplus \underbrace{L(v_2)}_{\in W}$ RHS

? ✓ 2) $L(\alpha \odot v) = \alpha \boxdot \underbrace{L(v)}_{\in W}$ $\alpha \in \mathbb{R}$

$L : \mathbb{R}^m \to \mathbb{R}^n$

$L : \mathbb{R}^n \to \mathbb{R}^{t \times s}$

Ex $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$L(\underbrace{(x,y)}_{\text{input}}) = (\underbrace{x+y, x-y, 0}_{\text{output}}) \in \mathbb{R}^3$

$(x, y) \longmapsto (x+y, x-y, 0)$

Ex

$L : \mathbb{R} \longrightarrow \mathbb{R}$

$\underbrace{(x,y)}_{\text{inputs}} \xmapsto{\text{operation!}} (x+y,\ x-y,\ 0)$

$L(\underbrace{(x,y)}_{\substack{\text{input}\\ \in \mathbb{R}^2}}) = (\underbrace{x+y,\ x-y,\ 0}_{\text{output}}) \in \mathbb{R}$

Is L a linear transformation?

1) LHS $L\left((x_1,y_1) + (x_2,y_2)\right) = L\left(\underbrace{(x_1+x_2,\ y_1+y_2)}_{\in \mathbb{R}^2}\right) = (\underbrace{x_1+x_2+y_1+y_2}_{\checkmark},\ \underbrace{x_1+x_2-y_1-y_2}_{\checkmark},\ \underbrace{0}_{\checkmark})$

RHS $\underbrace{L((x_1,y_1))}_{\in \mathbb{R}^3} + \underbrace{L((x_2,y_2))}_{\in \mathbb{R}^3} = (x_1+y_1,\ x_1-y_1,\ 0) + (x_2+y_2,\ x_2-y_2,\ 0) = (\underbrace{x_1+y_1+x_2+y_2}_{\checkmark},\ \underbrace{x_1-y_1+x_2-y_2}_{\checkmark},\ \underbrace{0+0}_{\checkmark})$

$\Big\} = \checkmark$

$\alpha \in \mathbb{R}$

2) LHS $L(\alpha(x,y)) = L((\alpha x, \alpha y)) = (\underbrace{\alpha x+\alpha y}_{\checkmark},\ \underbrace{\alpha x-\alpha y}_{\checkmark},\ \underbrace{0}_{\checkmark})$

RHS $\alpha \cdot L((x,y)) = \alpha \cdot (x+y,\ x-y,\ 0) = (\underbrace{\alpha(x+y)}_{\checkmark},\ \underbrace{\alpha(x-y)}_{\checkmark},\ \underbrace{\alpha.0}_{\checkmark})$

$\Big\} = \checkmark$

$\Rightarrow$ L is a linear transformation

Ex

$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ Is L a linear transformation?

$(x,y,z) \longmapsto (x+y,\ y+z-3)$

1) LHS $L\left((x_1,y_1,z_1)+(x_2,y_2,z_2)\right) = L\left((x_1+x_2,\ y_1+y_2,\ z_1+z_2)\right) = (\underbrace{x_1+x_2+y_1+y_2}_{\checkmark},\ y_1+y_2+z_1+z_2-3) \in \mathbb{R}^2$

RHS $L((x_1,y_1,z_1)) + L((x_2,y_2,z_2)) = (\underbrace{x_1+y_1,\ y_1+z_1-3}_{\in \mathbb{R}^2}) + (\underbrace{x_2+y_2,\ y_2+z_2-3}_{\in \mathbb{R}^2}) = (\underbrace{x_1+y_1+x_2+y_2}_{\checkmark},\ \underbrace{y_1+z_1-3+y_2+z_2-3}_{-6}) \in \mathbb{R}^2$

$\Big\} \neq$

$\Rightarrow$ L is NOT a linear transformation.

Ex

$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^{2\times 2}$ Is L a linear transformation?

$(x,y,z) \longmapsto \begin{bmatrix} x+y & x-y \\ y+z & y-z \end{bmatrix}_{2\times 2}$

1) $L\left((x_1,y_1,z_1)+(x_2,y_2,z_2)\right) = L\left((x_1+x_2,\ y_1+y_2,\ z_1+z_2)\right) = \begin{bmatrix} \underbrace{x_1+x_2+y_1+y_2}_{\checkmark} & \underbrace{x_1+x_2-y_1-y_2}_{\checkmark} \end{bmatrix}$

1) $L\left((x_1,y_1,z_1)+(x_2,y_2,z_2)\right) = L\left((\underline{x_1+x_2}, \underline{y_1+y_2}, \underline{z_1+z_2})\right) = \begin{bmatrix} \underset{\checkmark}{\underline{x_1+x_2+y_1+y_2}} & \underset{\checkmark}{\underline{x_1+x_2-y_1-y_2}} \\ \underset{\checkmark}{\underline{y_1+y_2+z_1+z_2}} & \underset{\checkmark}{\underline{y_1+y_2-z_1-z_2}} \end{bmatrix}_{2\times 2}$

$\underset{\checkmark}{\underline{L((x_1,y_2,z_2))}} + \underline{L((x_2,y_2,z_2))} = \begin{bmatrix} x_1+y_1 & x_1-y_1 \\ y_1+z_1 & y_1-z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 & x_2-y_2 \\ y_2+z_2 & y_2-z_2 \end{bmatrix} = \begin{bmatrix} \underset{\checkmark}{\underline{x_1+y_1+x_2+y_2}} & \underset{\checkmark}{\underline{x_1-y_1+x_2-y_2}} \\ \underset{\checkmark}{\underline{y_1+z_1+y_2+z_2}} & \underset{\checkmark}{\underline{y_1-z_1+y_2-z_2}} \end{bmatrix}_{2\times 2}$

$= \checkmark$

2) $L\left(\alpha(x,y,z)\right) = L\left((\alpha x, \alpha y, \alpha z)\right) = \begin{bmatrix} \underline{\alpha x+\alpha y} & \underline{\alpha x-\alpha y} \\ \underline{\alpha y+\alpha z} & \underline{\alpha y-\alpha z} \end{bmatrix}$

$\alpha \cdot \underline{L((x,y,z))} = \alpha \cdot \begin{bmatrix} x+y & x-y \\ y+z & y-z \end{bmatrix} = \begin{bmatrix} \alpha\underline{(x+y)} & \alpha\underline{(x-y)} \\ \alpha\underline{(y+z)} & \alpha\underline{(y-z)} \end{bmatrix}$

$= \checkmark$

$\Rightarrow$ L is a linear transformation.